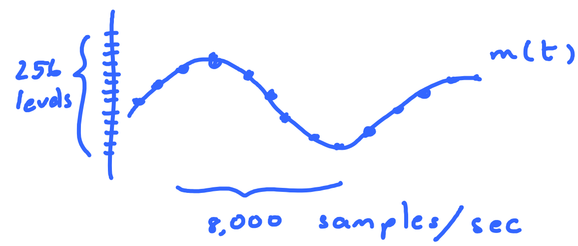
8 Sampling and Aliasing

Wednesday, August 22, 2012
8:50 AM

[Slides] : Digitization (two major step)

- sampling ← This is the focus of this section
- quantization

Ex. telephone signal (voice)



speech/voice → 3.5 kHz

Sample @ 8,000 samples/sec

256 levels → 8 bits per sample

$256 = 2^8$

$$8 \frac{\text{bits}}{\cancel{\text{sample}}} \times 8000 \, \cancel{\text{samples}}/\text{sec} = 64 \text{ kbps}$$

Ex. CD audio (Red book standard for CD audio)

audio : 20 kHz max. freq.

Sampling rate : 44.1 k samples/sec

2^{16} quantization levels → 16 bits/sample

(16-bit two's complement integer)

$$\underbrace{2}_{\text{left/right audio channel}} \times 44.1 \text{ k} \frac{\text{Sa}}{\text{s}} \times 16 \frac{\text{b}}{\text{Sa}} \times 80 \frac{\text{min}}{\text{CD}} \times 60 \frac{\text{s}}{\text{min}} \times \frac{1}{8} \frac{\text{B}}{\text{b}} \approx 800 \frac{\text{MB}}{\text{CD}}$$

→ 1.4112 Mbps (audio bit rate)

Sampling

continuous-time signal		discrete-time signal
$m(t)$, $-\infty < t < \infty$	$\longrightarrow$	$\ldots, m[-2], m[-1], m[0], m[1], m[2], \ldots$

$m(t) \longrightarrow \ldots, m[-2], m[-1], m[0], m[1], m[2], \ldots$

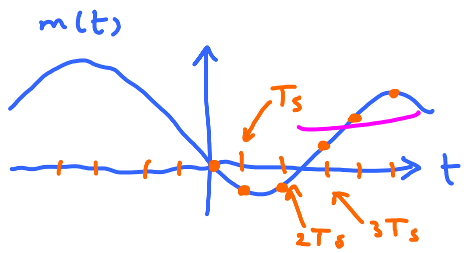
$-\infty < t < \infty$

$$m[k] = m(kT_s) = m(t)\Big|_{t=kT_s}$$

T_s: sampling interval

Remark: Writing f_s as R_s may be more appropriate and better match with the name sampling "rate"

$$f_s = \frac{1}{T_s} = \text{sampling rate}$$

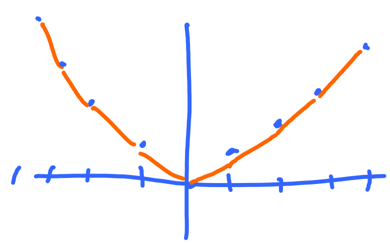


Note: at this step, we assume "infinite precision" for each value of $m[k]$.

Examples:

Plot $y = x^2$ $\Longrightarrow$ ① "Evaluate" $y = x^2$ at many values of x

② "connect" the dots



Step ① is sampling
and step ② is reconstruction

Plot $y = \sin(100\pi t)$ [see slides]

- need f_s to be large enough
- otherwise, see aliasing.

Theory behind Sampling theorem

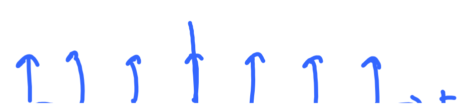
Here, we use $g(t)$ to denote the signal under consideration.

You may replace $g(t)$ below by $m(t)$ if you are thinking about $g(t)$ as a message to be transmitted by a communication system.
We use $g(t)$ here because the results provided below work in broader setting as well.

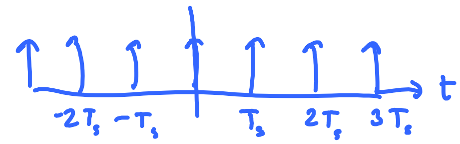
① Ideal Sampling

First, recall the shah function and its Fourier transform:

$$Ш(t) = \sum^{\infty} \delta(t - kT_s)$$

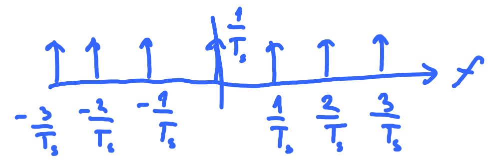


$$Ш_{T_s}(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$$



$$\Big\downarrow \mathcal{F}$$

$$\mathcal{F}\{Ш_{T_s}(t)\} = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{T_s}\right)$$



$$= f_s \sum_{k=-\infty}^{\infty} \delta(f - kf_s)$$

$$= f_s \, Ш_{f_s}(f)$$

The (ideal instantaneous) sampled signal is given by

$$g_\delta(t) = g(t) \times Ш_{T_s}(t) = g(t) \times \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$$

$$= \sum_k g(t)\delta(t - kT_s) = \sum_k \underbrace{g(kT_s)}_{g[k]} \delta(t - kT_s)$$

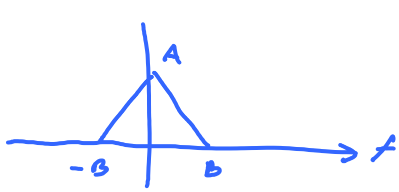
$$\mathcal{F}$$

$$G_\delta(f) = G(f) * \left(Ш_{f_s}(f)\right) = G(f) * \left(f_s \sum_k \delta(f - kf_s) \right)$$

$$= f_s \sum_k G(f - kf_s)$$

As usual, we will assume that the signal $g(t)$ is bandlimited to B.

$|G(f)|$

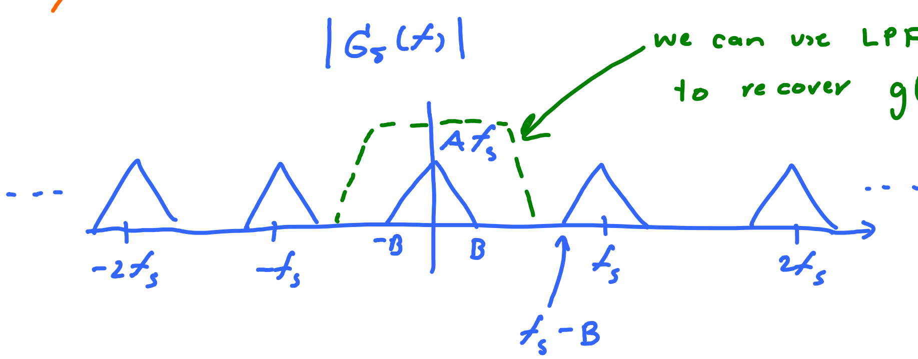


In which case, the Fourier transform of the sampled signal is given by

$|G_\delta(f)|$

we can use LPF with gain $\frac{1}{f_s}$

is given by



To prevent the corruption of the original signal (in the freq.), we need

$$f_s - B > B$$

$$f_s > 2B \leftarrow \text{Sampling theorem}$$

Observation : (i) $\mathcal{F}\{g_\delta(t)\} = G_\delta(f)$ is "periodic" with "period" f_s.

$\Rightarrow$ plotspec only shows the freq. from $-\frac{f_s}{2}$ to $\frac{f_s}{2}$

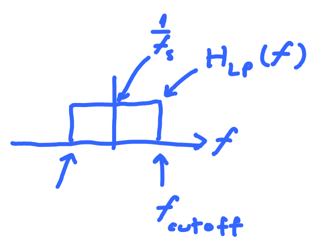
(ii) Even though this sampling is "ideal" because it involves the use of δ - function.
We can get a useful result from the reconstruction part.

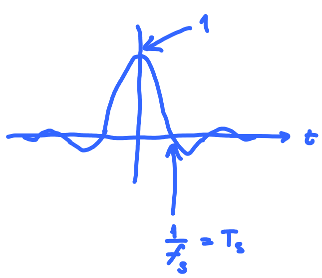
② ideal reconstruction

$$g_\delta(t) \longrightarrow \boxed{\text{LPF}} \longrightarrow g(t)$$

LPF

$h_{LP}(t) \xrightarrow{\mathcal{F}}$





This will work as long as

$$B < f_{cutoff} < f_s - B$$

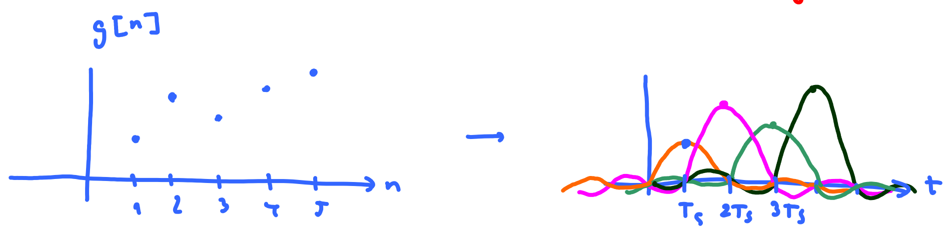
We choose $f_{cutoff} = \frac{f_s}{2}$

$$h_{LP}(t) = \text{sinc}(\pi f_s t)$$

$$g(t) = g_\delta(t) * h_{LP}(t) = \left(\sum_n g[n]\,\delta(t - nT_s)\right) * h_{LP}(t)$$

$$= \sum_n g[n]\, h_{LP}(t-nT_s) = \sum_n g[n]\, \mathrm{sinc}(\pi f_s (t-nT_s))$$

Reconstruction eqn.



Remark: The big picture:

$g(t)$ is a continuous-time signal.

$g_\delta(t) = g(t) \times Ш_{T_s}(t)$ is also a continuous-time signal.

However, $g_\delta(t)$ is 0 all the time except at kT_s where we have weigthed δ-function.

We define $g_\delta(t)$ so that we can have an easy way to analyze $g[n]$ below.

(Another approach is to use DTFT.)

It provides an intermediate step that leads to the sampling theorem, the Nyquist sampling rate requirement, and the reconstruction eqn.

It also provides a way to "see" aliasing.

$g[n]$ is a discrete-time signal.

This is simply a sequence of numbers.

The reconstruction eqn. says that we can recover $g(t)$ back from $g[n]$.

So, there is no need to transmit the whole signal $g(t)$.

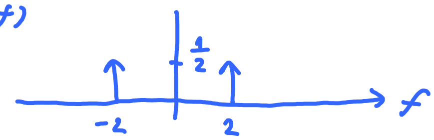
We only need to transmit $g[n]$.

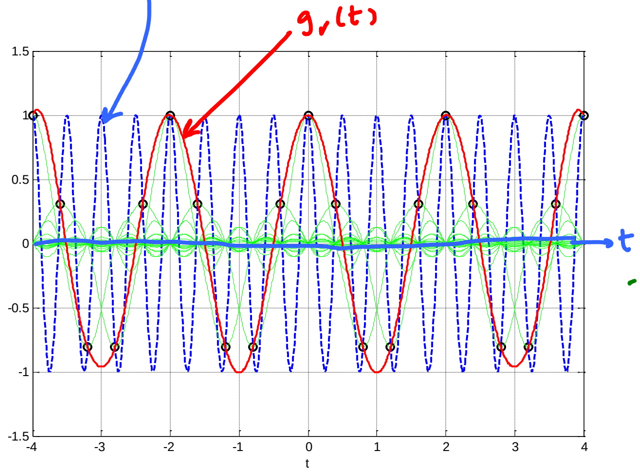
Sampling Theorem

If a signal $g(t)$ contains no freq. content for $|f| \ge B$, it is completely described by instantaneous sampled values uniformly spaced in time with period $T_s \le 1/2B$.

In which case, $g(t)$ can be reconstructed from its samples $(\dots, g[-2], g[-1], g[0], g[1], g[2], \dots)$ by the reconstruction equation above equation above

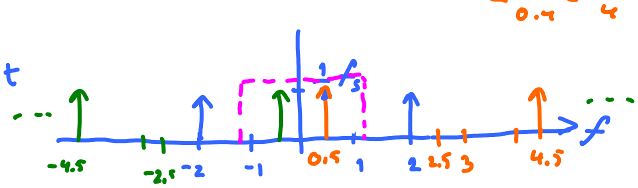
Ex. $\cos(2\pi\, 2t) \equiv g(t) \xrightarrow{\mathcal{F}} G(f)$





$$G_\delta(f) = f_s \sum_k G(f - k f_s)$$

$\uparrow \frac{1}{0.4} = \frac{10}{4} = 2.5$



$g_r(t) = \cos(2\pi\, 0.5\, t)$

Of course, using AM, FM, PM that we studied earlier, we can transmit $g(t)$ using carrier $\cos(2\pi f_c t)$.
However, here we will transmit it via different technique(s).